Aequat. Math. 90 (2016), 133–145

0001-9054/16/010133-13
*published online* October 13, 2015
DOI 10.1007/s00010-015-0376-9

**Aequationes Mathematicae**

# Additive iterative roots of identity and Hamel bases

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*Dedicated to Professor Roman Ger on his 70th birthday.*

**Abstract.** We describe a class of discontinuous additive functions $a : X \to X$ on a real topological vector space $X$ such that $a^n = \mathrm{id}_X$ and $a(\mathcal{H})\backslash\mathcal{H} \neq \emptyset$ for every infinite set $\mathcal{H} \subset X$ of vectors linearly independent over $\mathbb{Q}$. We prove the density of the family of all such functions in the linear topological space $\mathcal{A}_X$ of all additive functions $a : X \to X$ with the topology induced on $\mathcal{A}_X$ by the Tychonoff topology of the space $X^X$. Moreover, we consider additive functions $a \in \mathcal{A}_X$ satisfying $a^n = \mathrm{id}_X$ and $a(\mathcal{H}) = \mathcal{H}$ for some Hamel basis $\mathcal{H}$ of $X$. We show that the class of all such functions is also dense in $\mathcal{A}_X$. The method is based on decomposition theorems for linear endomorphisms.

**Mathematical Subject Classification.** Primary 39B52, 47A62.

**Keywords.** Additive function, Hamel base, Babbage's equation, Iterative root.

## 1. Introduction

For an integer $n \geq 0$ and arbitrary $T : X \to X$ by $T^n$ we denote the $n$-th *iterate* of $T$, i.e. the function $T^n : X \to X$ defined recursively by

$$\begin{cases} T^0 = \mathrm{id}_X, \\ T^k = T \circ T^{k-1} \quad \text{for } k \geq 1, \end{cases}$$

where $\mathrm{id}_X : X \to X$ is the identity map and $\circ$ is the composition of functions.

A function $a : X \to X$ defined on a real vector space $X$ is called *additive* provided $a(x + y) = a(x) + a(y)$ for $x, y \in X$. Put

$$\mathcal{A}_X = \{a \in X^X : a \text{ is additive}\}.$$

By a *Hamel basis* of a real vector space $X$ we mean an arbitrary algebraic basis of the space $X$ over $\mathbb{Q}$. It is known that every additive mapping $a \in \mathcal{A}_X$ is $\mathbb{Q}$-linear (a proof of this simple fact for $X = \mathbb{R}^n$ may be found in [3] and it is valid in the general case for real vector spaces). Hence every additive function $a \in \mathcal{A}_X$ can be considered as a linear operator on the space $X$ over the field $\mathbb{Q}$.

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If $X$ is a real topological vector space then we treat $X^X$ as a topological vector space of all functions $f : X \to X$ with the usual addition and multiplication by scalars and with the Tychonoff topology. We consider then $\mathcal{A}_X$ as a subspace of $X^X$.

Let $X$ be a nontrivial real topological vector space. K. Baron constructed in [1] an example of a discontinuous additive function $a \in \mathcal{A}_X$ satisfying $a^2 = \mathrm{id}_X$ ($a$ is called then an *involution*) and $a(\mathcal{H})\backslash\mathcal{H} \neq \emptyset$ for every at least three-element set $\mathcal{H} \subset X$ of vectors linearly independent over $\mathbb{Q}$. It is also proved there that the sets

$$\{a \in \mathcal{A}_X : a \text{ is a discontinuous involution and } a(\mathcal{H})\backslash\mathcal{H} \neq \emptyset \text{ holds for each uncountable and linearly independent over } \mathbb{Q} \text{ set } \mathcal{H} \subset X\},$$

$$\{a \in \mathcal{A}_X : a \text{ is a discontinuous involution and } a(\mathcal{H}) = \mathcal{H} \text{ for a Hamel basis } \mathcal{H} \text{ of the space } X \text{ over the field } \mathbb{Q}\}$$

are dense in $\mathcal{A}_X$. Simple algebraic proofs of these results are given in [2].

K. Baron presented results [1, Theorem 2] during the 15th International Conference on Functional Equations and Inequalities (Ustroń, May 19–25, 2013). Following the question posed by J. Schwaiger after Baron's lecture we solve

**Problem 1.** (*J. Schwaiger*) Is it possible to prove similar results for additive functions satisfying $a^n = \mathrm{id}$, where $n \geq 3$ is an integer?

If $X$ is a nonempty set then by an *iterative root of order* $n \geq 2$ *of the identity* we mean a solution $f : X \to X$ of Babbage's equation

$$f^n = \mathrm{id}_X.$$

Let $n \geq 2$ be an integer and put

$$\mathcal{A}_X^{(n)} = \{a \in \mathcal{A}_X : a^n = \mathrm{id}_X\}.$$

Our aim is to prove that the sets

$$\{a \in \mathcal{A}_X^{(n)} : a \text{ is discontinuous and } a(\mathcal{H})\backslash\mathcal{H} \neq \emptyset \text{ for every infinite set } \mathcal{H} \subset X \text{ of vectors linearly independent over } \mathbb{Q}\},$$

$$\{a \in \mathcal{A}_X^{(n)} : a \text{ is discontinuous and } a(\mathcal{H}) = \mathcal{H} \text{ for a Hamel basis } \mathcal{H} \text{ of the space } X \text{ over the field } \mathbb{Q}\}$$

are dense in $\mathcal{A}_X$. Our solution is based on structural results concerning the equation

$$p(T) = 0, \tag{1}$$

where $p \in \mathbb{K}[x]$ is a polynomial with $\deg p > 0$ and $T : V \to V$ is a linear operator on a vector space $V$ over a field $\mathbb{K}$. Then $p$ is called an *annihilating polynomial* for $T$. For the convenience of the reader we recall useful facts and statements from [4, Section 2].

## 2. Annihilating polynomials and decomposition on cyclic spaces

Let $V$ be a vector space over a field $\mathbb{K}$ and let $T : V \to V$ be a linear operator. For any $0 \neq v \in V$ let us denote by

$$\mathbf{Z}(v, T) = \operatorname{span}\left\{v, T(v), T^2(v), \dots\right\}$$

the so called $T$-*cyclic* subspace of $V$ generated by $v$. The restriction $T|_{\mathbf{Z}(v,T)}$ is a linear operator on $\mathbf{Z}(v, T)$ and it will be denoted by $T_v$. If $k$ is the lowest positive integer such that $\left\{v, T(v), \dots, T^k(v)\right\}$ is linearly dependent then $\dim \mathbf{Z}(v, T) = k$ and we find $\lambda_{k-1}, \dots, \lambda_1, \lambda_0 \in \mathbb{K}$ such that

$$T^k(v) = -\lambda_{k-1}T^{k-1}(v) - \cdots - \lambda_1 T(v) - \lambda_0 v.$$

Then $m_v(x) = x^k + \lambda_{k-1}x^{k-1} + \cdots + \lambda_1 x + \lambda_0 \in \mathbb{K}[x]$ is the *minimum polynomial* for $T_v$.

Further on, if $U$ is a vector space of finite dimension $n \geq 1$ over $\mathbb{K}$, then for every ordered basis $\mathcal{B} = (u_1, \dots, u_n)$ the linear operator $T : U \to U$ defined by

$$\begin{cases} T(u_j) = u_{j+1} & \text{for } j \in \{1, \dots, n-1\}, \\ T(u_n) = -\lambda_{n-1}u_n - \cdots - \lambda_1 u_2 - \lambda_0 u_1, & \end{cases}$$

has $p(x) = x^n + \lambda_{n-1}x^{n-1} + \cdots + \lambda_1 x + \lambda_0 \in \mathbb{K}[x]$ as its characteristic polynomial. This operator is called the $(p, \mathcal{B})$-*induced operator* on $U$. Then we have $U = \mathbf{Z}(u_1, T)$ and $p = m_{u_1}$.

Let $(W_i)_{i\in I}$ be a family of subspaces of a space $W$. The space $W$ is called a *direct sum* of $(W_i)_{i\in I}$ if every $w \in W$ can be written uniquely as $w = \sum_{i\in I} w_i$, where $w_i \in W_i$ for $i \in I$ and all but a finite number $w_i$ are zero. We write then $W = \bigoplus_{i\in I} W_i$. Assume moreover that for all $i \in I$ linear operators $T_i : W_i \to W_i$ are defined. By $\bigoplus_{i\in I} T_i$ we denote a unique linear operator $T : W \to W$ such that $T|_{W_i} = T_i$ for every $i \in I$. The description of linear operators satisfying (1) uses the following structural results.

**Proposition 1.** (Primary canonical decomposition, [4, Theorem 7]) *Let $V$ be a non-trivial vector space over $\mathbb{K}$ and let $T$ be a linear operator on $V$ with a monic annihilating polynomial $p \in \mathbb{K}[x]$ of positive degree which has a decomposition*

$$p = p_1 \cdot \ldots \cdot p_k, \tag{2}$$

*where $p_j \in \mathbb{K}[x]$ for $j \in \{1, \dots, k\}$ are relatively prime monic factors. Then*

$$V = \bigoplus_{j=1}^{k} \ker p_j(T) = \bigoplus_{j=1}^{k} \{v \in V : p_j(T)(v) = 0\}, \tag{3}$$

$\ker p_j(T)$ *are invariant subspaces for $T$, each $T_j := T|_{\ker p_j(T)}$ has $p_j$ as an annihilating polynomial and $T = \bigoplus_{j=1}^{k} T_j$.*

*If, moreover, $V$ has finite dimension and $p$ is the minimum polynomial for $T$ with the decomposition* (2) *then* $\ker p_j(T)$ *are nontrivial invariant subspaces for $T$ in the decomposition* (3)*. Finally, if $p_j$ are irreducible over $\mathbb{K}$ additionally, then*

$$\ker p_j(T) = \bigoplus_{l=1}^{m_j} \mathbf{Z}(v_j^l, T) \quad \textit{for} \quad j \in \{1, \dots, k\},$$

$\mathbf{Z}(v_j^l, T)$ *are $T$-cyclic subspaces with* $\dim \mathbf{Z}(v_j^l, T) = \deg p_j$ *and* $T_j = \bigoplus_{l=1}^{m_j} T_{v_j^l}$.

**Proposition 2.** (General canonical decomposition, [4, Proposition 2]) *Let $V$ be a non-trivial vector space over $\mathbb{K}$ and let $T$ be a linear operator on $V$ with a monic annihilating polynomial $p \in \mathbb{K}[x]$ which has a decomposition*

$$p = q_1^{r_1} \cdot \ldots \cdot q_s^{r_s}, \tag{4}$$

*where $q_j \in \mathbb{K}[x]$ for $j \in \{1, \dots, s\}$ are distinct irreducible monic factors and $r_j$ are positive integers for $j \in \{1, \dots, s\}$. Then $V$ has a decomposition*

$$V = \bigoplus_{i \in I} Z_i, \tag{5}$$

*where $Z_i = \mathbf{Z}_i(v_i, T)$ for $i \in I$ are $T$-cyclic and $T_i := T|_{Z_i}$ for $i \in I$ have minimum polynomials $q_j^l$ for some $1 \le j \le s$ and $1 \le l \le r_j$.*

*Conversely, if $V$ admits a decomposition* (5) *and $T_i$ on $Z_i$ are $q_j^l$-induced, then $T := \bigoplus_{i \in I} T_i$ has $p$ given by* (4) *as an annihilating polynomial.*

Finally, if $\mathcal{B}$ is a basis of $V$ and $n$ is a positive integer, then a family $(\mathcal{B}_i)_{i \in I}$ is called an *$n$-partition* of $\mathcal{B}$ if $\bigcup_{i \in I} \mathcal{B}_i = \mathcal{B}$, $\operatorname{card} \mathcal{B}_i = n$ for every $i \in I$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for all $i, j \in I$ such that $i \neq j$. The family $(\mathcal{B}_i)_{i \in I}$ will also be called an *$n$-partitioned basis* of $V$. If every $\mathcal{B}_i$ is ordered by writing $\mathcal{B}_i = \left(v_1^i, \dots, v_n^i\right)$, then we call $(\mathcal{B}_i)_{i \in I}$ an *$n$-partitioned ordered basis* of $V$. In this case elements $v_1^i$ are called the *lead elements* in a partitioned ordered basis.

The following corollary is an interpretation of Proposition 2.

**Corollary 1.** (cf. [4, Proposition 2]) *Let $V$ be a non-trivial vector space over $\mathbb{K}$ and let $T$ be a linear operator on $V$ with an annihilating monic polynomial $p \in \mathbb{K}[x]$ which has a decomposition* (4)*, where $q_j \in \mathbb{K}[x]$ for $j \in \{1, \dots, s\}$ are distinct irreducible monic factors with* $\deg q_j = k_j$ *and $r_j$ are positive integers for $j \in \{1, \dots, s\}$.*

*Then there exists a basis $\mathcal{B}$ of $V$ over $\mathbb{K}$, its ordered partition $(\mathcal{B}_i)_{i \in I}$ into subsets having cardinality $lk_j$ for some $1 \le j \le s$ and $1 \le l \le r_j$, such that $T = \bigoplus_{i \in I} T_i$, where for every $\mathcal{B}_i$ of cardinality $lk_j$ (as it has been chosen in the partition) the operator $T_i$ is $(q_j^l, \mathcal{B}_i)$-induced.*

*Conversely, if we choose a basis $\mathcal{B}$ of $V$ over $\mathbb{K}$, an ordered partition $(\mathcal{B}_i)_{i \in I}$ of $\mathcal{B}$ into subsets having cardinality $lk_j$ for some $1 \le j \le s$ and $1 \le l \le r_j$, and for each $\mathcal{B}_i$ of cardinality $lk_j$ (as it has been chosen in the partition) we define the $(q_j^l, \mathcal{B}_i)$-induced operator $T_i$, then the operator $T = \bigoplus_{i \in I} T_i$ satisfies* (1).

## 3. Primary decomposition spaces

Let $X$ be a real vector space. To every $a \in \mathcal{A}_X^{(n)}$ we associate a decomposition of the space $X$. This decomposition is crucial for further considerations.

**Lemma 1.** *For every $a \in \mathcal{A}_X^{(n)}$ there is a unique decomposition of the space $X$ over $\mathbb{Q}$ of the form $X = X_a^+ \oplus X_a^-$, where*

$$X_a^+ = \{v \in X : a(v) = v\} \quad \text{and} \quad X_a^- = \left\{ v \in X : \sum_{j=0}^{n-1} a^j(v) = 0 \right\}$$

*are invariant subspaces of $X$ over $\mathbb{Q}$ with respect to $a$.*

*Proof.* Note that $x^n - 1 = (x-1)\sum_{j=0}^{n-1} x^j$. Since $(x-1)$ and $\sum_{j=0}^{n-1} x^j$ are relatively prime over $\mathbb{Q}$, the decomposition $X = X_a^+ \oplus X_a^-$ is a consequence of Proposition 1. □

Fix $a \in \mathcal{A}_X^{(n)}$. Considering the dimensions of the spaces $X_a^+$ and $X_a^-$ over $\mathbb{Q}$ we will be interested only in distinguishing whether these dimensions are finite or not. Put

$$\mathcal{A}_X^{(n),+} = \left\{ a \in \mathcal{A}_X^{(n)} : 1 \le \dim_{\mathbb{Q}} X_a^+ < \infty \right\},$$
$$\mathcal{A}_X^{(n),-} = \left\{ a \in \mathcal{A}_X^{(n)} : 1 \le \dim_{\mathbb{Q}} X_a^- < \infty \right\}.$$

**Theorem 1.** *Let $X$ be a nontrivial real topological vector space. Then each additive function $a \in \mathcal{A}_X^{(n),+} \cup \mathcal{A}_X^{(n),-}$ is discontinuous.*

*Proof.* Fix $a \in \mathcal{A}_X^{(n)}$ and suppose that $a$ is continuous. Then $a$ is $\mathbb{R}$-linear (for the proof of this simple fact for $X = \mathbb{R}^n$ we refer the Reader to [3] and its generalization to real topological vector spaces can be proved in the same way). Hence both, $X_a^+$ and $X_a^-$ are closed subspaces of $X$. If $a \in \mathcal{A}_X^{(n),+}$ ($a \in \mathcal{A}_X^{(n),-}$, respectively) then $X_a^+$ ($X_a^-$, respectively) is a nontrivial closed subspace, which has finite dimension over $\mathbb{Q}$. This contradiction proves that $a$ is discontinuous. □

Examples of members of classes $\mathcal{A}_X^{(p),+}$ and $\mathcal{A}_X^{(p),-}$ for prime numbers $p$ will be given later when we prove the description of functions belonging to $\mathcal{A}_X^{(p)}$.

## 4. Description of additive functions in $\mathcal{A}_X^{(p)}$

Let $X$ be a real vector space and let $p$ be a prime number. Our description is based on the decomposition $q(x) = x^p - 1 = (x-1)\sum_{j=0}^{p-1} x^j$. Note that for any prime number $p$, the polynomials $q_1(x) = x - 1$ and $q_2(x) = \sum_{j=0}^{p-1} x^j$ are relatively prime and irreducible over $\mathbb{Q}$. We begin with the general description of elements of $\mathcal{A}_X^{(p)}$.

**Theorem 2.** *If $p$ is a prime number and $a \in \mathcal{A}_X^{(p)}$ then*

(i) *$X$ has a decomposition*

$$X = X_a^+ \oplus \bigoplus_{i\in I} \mathbf{Z}_i(v_i, a) \tag{6}$$

*where $\mathbf{Z}_i(v_i,a)$ for $i \in I$ is $a$-cyclic and $a_i := a|_{\mathbf{Z}_i(v_i,a)}$ has $q_2$ as a minimum polynomial;*

(ii) *there exists a Hamel basis $\mathcal{B}$ of $X$ having a partition $\mathcal{B} = \overline{\mathcal{B}} \cup \bigcup_{i\in I} \mathcal{B}_i$ with $\operatorname{card} \mathcal{B}_i = p-1$, such that $a = \overline{a} \oplus \bigoplus_{i\in I} a_i$, where $\operatorname{span} \overline{\mathcal{B}} = X_a^+$, $\overline{a} = a|_{X_a^+} = \mathrm{id}_{X_a^+}$, and for every $i \in I$ the additive function $a_i$ is $(q_2, \mathcal{B}_i)$-induced.*

*Conversely, for a Hamel basis $\mathcal{B} \subset X$ and for its ordered partition $\mathcal{B} = \overline{\mathcal{B}} \cup \bigcup_{i\in I} \mathcal{B}_i$ with $\operatorname{card} \mathcal{B}_i = p-1$ let $\overline{a} = \mathrm{id}_{\operatorname{span} \overline{\mathcal{B}}}$ and let additive functions $a_i$ be $(q_2, \mathcal{B}_i)$-induced for every $i \in I$. Then $a = \overline{a} \oplus \bigoplus_{i\in I} a_i \in \mathcal{A}_X^{(p)}$. In this case $\operatorname{span} \overline{\mathcal{B}} = X_a^+$ and $\operatorname{span} \left(\bigcup_{i\in I} \mathcal{B}_i\right) = X_a^-$.*

*Proof.* Fix $a \in \mathcal{A}_X^{(n)}$. By Lemma 1 the space $X$ admits a decomposition $X = X_a^+ \oplus X_a^-$ into invariant subspaces $X_a^+$ and $X_a^-$ over $\mathbb{Q}$. Since $q_2$ is irreducible, on account of Proposition 2 the space $X_a^-$ has a decomposition

$$X_a^- = \bigoplus_{i\in I} \mathbf{Z}_i(v_i, a),$$

where $\mathbf{Z}_i(v_i,a)$ for $i \in I$ are $a$-cyclic and each $a_i := a|_{\mathbf{Z}_i(v_i,a)}$ has $q_2$ as a minimum polynomial. Thus $X$ has a decomposition (6).

To prove (ii) put $\overline{a} := a|_{X_a^+}$. Let $\overline{\mathcal{B}} = (b_j)_{j\in J}$ be an arbitrary basis of $X_a^+$. For each $i \in I$ let $\mathcal{B}_i = (a^k(v_i))_{k\in\{0,\dots,p-2\}}$ be an ordered basis of $\mathbf{Z}_i(v_i,a)$. Then $a_i = a|_{\mathbf{Z}_i(v_i,a)}$ is $(q_2, \mathcal{B}_i)$-induced. By (6) the family $\mathcal{B} := \overline{\mathcal{B}} \cup \bigcup_{i\in I} \mathcal{B}_i$ is then a Hamel basis of $X$. Finally, $a = \overline{a} \oplus \bigoplus_{i\in I} a_i$.

The converse is a refinement of Corollary 1. $\square$

*Example* 1. Let $\mathcal{B} \subset X$ be a Hamel basis. For a positive integer $k$ choose $\mathcal{B}' = \{b_1, \dots, b_k\} \subset \mathcal{B}$, and next let $(\mathcal{B}_i)_{i\in I}$ be an ordered $(p-1)$-partition of $\mathcal{B} \backslash \mathcal{B}'$. Finally, let $a_0 = \mathrm{id}_{\operatorname{span} \mathcal{B}'}$ and for each $i \in I$ let $a_i$ be $(q_2, \mathcal{B}_i)$-induced on $\operatorname{span} \mathcal{B}_i$. Then $a = \left(a_0 \oplus \bigoplus_{i\in I} a_i\right) \in \mathcal{A}_X^{(p),+}$.

*Example* 2. Let $\mathcal{B} \subset X$ be a Hamel basis. For a positive integer $k$ let $(\mathcal{B}_i)_{i\in\{1,\dots,k\}}$ be a family of disjoint ordered subsets $\mathcal{B}_i \subset \mathcal{B}$ with $\operatorname{card} \mathcal{B}_i = p-1$ for $i \in \{1, \dots, k\}$. Put $a_0 = \mathrm{id}_{\operatorname{span}\left(\mathcal{B} \backslash \bigcup_{i=1}^k \mathcal{B}_i\right)}$ and let $a_i$ be $(q_2, \mathcal{B}_i)$-induced on $\operatorname{span} \mathcal{B}_i$ for every $i \in \{1, \dots, k\}$. Then $a = \left(a_0 \oplus \bigoplus_{i=1}^k a_i\right) \in \mathcal{A}_X^{(p),-}$.

## 5. Properties of elements of the classes $\mathcal{A}_X^{(n),+}$ and $\mathcal{A}_X^{(n),-}$

We collect here some properties of elements of the families $\mathcal{A}_X^{(n),+}$ and $\mathcal{A}_X^{(n),-}$. We will first deal with the class $\mathcal{A}_X^{(n),+}$. We begin with the obvious

**Lemma 2.** *Let $X$ be a nonempty set and let $f : X \to X$ be a function satisfying $f^n = \mathrm{id}_X$. Then for every $x \in X$ the orbit $(f^k(x))_{k\geq 1}$ is a cycle with the length being a divisor of $n$.*

**Lemma 3.** *Let $a \in \mathcal{A}_X^{(n)}$ and let $\mathbf{v} \subset X$ be a finite set of vectors linearly independent over $\mathbb{Q}$. If $a(\mathbf{v}) = \mathbf{v}$, then $\dim_{\mathbb{Q}} X_a^+ \geq \left[\frac{\operatorname{card} \mathbf{v}}{n}\right]$, where $[x]$ denotes the integer part of $x$.*

*Proof.* We may write $\mathbf{v} = \bigcup_{l=1}^{q} \mathbf{v}_l$, where each $\mathbf{v}_l$ is an orbit in $\mathbf{v}$ (with respect to $a|_{\mathbf{v}} : \mathbf{v} \to \mathbf{v}$), which by Lemma 2 is a cycle of the length, let us say $d_l$, being a divisor of $n$. Hence we have

$$\operatorname{span} \mathbf{v} = \bigoplus_{l=1}^{q} \operatorname{span} \mathbf{v}_l. \tag{7}$$

Choose vectors $v_l \in \mathbf{v}_l$ in the following way. If $d_l = 1$, then take $v_l$ as the unique element of the orbit $\mathbf{v}_l$. If $d_l \geq 2$, let us say $\mathbf{v}_l = \{w_1, \ldots, w_{d_l}\}$, then clearly

$$a(w_1) = w_2, \ \ldots, \ a(w_{d_l}) = w_1. \tag{8}$$

Put $v_l = w_1 + \cdots + w_{d_l}$. Since $w_1, \ldots, w_{d_l}$ are linearly independent over $\mathbb{Q}$, $v_l \neq 0$.

We claim that $v_1, \ldots, v_q$ are linearly independent and $a(v_l) = v_l$ for every $l \in \{1, \ldots, q\}$. Indeed, since $v_l \neq 0$ and $v_l \in \operatorname{span} \mathbf{v}_l$, by (7) the vectors $v_1, \ldots, v_q$ are linearly independent. Further on, for $d_l = 1$ we have $a(v_l) = v_l$ ($\{v_l\} = \mathbf{v}_l$ is in this case a one element orbit). If $d_l \geq 2$, then by (8) we get

$$a(v_l) = a(w_1 + \cdots + w_{d_l - 1} + w_{d_l}) = w_2 + \cdots + w_{d_l} + w_1 = v_l.$$

Since $a(v_l) = v_l$ for $l \in \{1, \ldots, q\}$ and vectors $v_1, \ldots, v_q$ are linearly independent, $\dim_{\mathbb{Q}} X_a^+ \geq q$. Finally, the number $q$ of orbits is at least equal to $\left[\frac{\operatorname{card} \mathbf{v}}{n}\right]$. □

**Corollary 2.** *If $a \in \mathcal{A}_X^{(n),+}$ then $a(\mathbf{v}) \backslash \mathbf{v} \neq \emptyset$ for every infinite set $\mathbf{v} \subset X$ of vectors linearly independent over $\mathbb{Q}$.*

*Proof.* Fix $a \in \mathcal{A}_X^{(n),+}$ with $\dim_{\mathbb{Q}} X_a^+ = r$ and suppose that $a(\mathbf{v}) \subset \mathbf{v}$ for an infinite set $\mathbf{v} \subset X$ of vectors linearly independent over $\mathbb{Q}$. Since $a^n = \mathrm{id}_X$, we find a family $(\mathbf{v}_l)_{l\in\{1,\ldots,r+1\}}$ of orbits such that $\mathbf{v}' = \bigcup_{l=1}^{r+1} \mathbf{v}_l \subset \mathbf{v}$ and $a(\mathbf{v}') = \mathbf{v}'$. For each orbit we find $v_l \in \operatorname{span} \mathbf{v}_l$ such that $a(v_l) = v_l$ for $l \in \{1, \ldots, r+1\}$. Since vectors $v_1, \ldots, v_{r+1}$ are linearly independent (cf. Proof of Lemma 3), we have $\dim_{\mathbb{Q}} X_a^+ \geq r + 1$. This contradiction finishes the proof. □

We will proceed now with the class $\mathcal{A}_X^{(n),-}$ in a particular case. Let $p$ be a prime number. We have

**Lemma 4.** *Let $a : U \to U$ be an additive function on a $p$-dimensional vector space $U$ over $\mathbb{Q}$. The following conditions are equivalent:*

(i) *there exists a basis $\mathbf{u} = (u_1, \dots, u_p)$ of the space $U$ over $\mathbb{Q}$ such that*

$$\begin{cases} a(u_j) = u_{j+1} \quad \text{for} \quad j \in \{1, \dots, p-1\}, \\ a(u_p) = u_1, \end{cases} \tag{9}$$

(ii) *there exists a basis $\mathbf{v} = (v_1, \dots, v_p)$ of $U$ such that*

$$\begin{cases} a(v_1) = v_1, \\ a(v_j) = v_{j+1} \qquad\qquad \text{for} \quad j \in \{2, \dots, p-1\}, \\ a(v_p) = -v_2 - \dots - v_p. \end{cases} \tag{10}$$

*Remark* 1. The statement of Lemma 4 can be equivalently formulated for matrices. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix},$$

be square matrices of dimension $p$. Then the matrices $\mathbf{A}$ and $\mathbf{A}'$ are similar over $\mathbb{Q}$, that is there exists a nonsingular matrix $\mathbf{C}$ over $\mathbb{Q}$ such that

$$\mathbf{CAC}^{-1} = \mathbf{A}'.$$

*Proof.* Let $\mathbf{u} = (u_1, \dots, u_p)$ be a Hamel basis of the space $U$ such that (9) holds. Then $a$ is $(q, \mathbf{u})$-induced on $U$ with the minimum polynomial $q(x) = x^n - 1 = q_1(x) \cdot q_2(x)$, where $q_1(x) = x - 1$ and $q_2(X) = \sum_{j=0}^{p-1} x^j$ are irreducible over $\mathbb{Q}$ and relatively prime. On account of Proposition 1 we have

$$U = \ker q_1(a) \oplus \ker q_2(a) = U_a^+ \oplus U_a^-, \tag{11}$$

where $U_a^+ = \ker q_1(a)$ and $U_a^- = \ker q_2(a)$ are nontrivial $a$-cyclic subspaces of $U$ with $\dim_{\mathbb{Q}} U_a^+ = 1$ and $\dim_{\mathbb{Q}} U_a^- = p - 1$. We find bases $(v_1)$ in $U_a^+$ and $(v_2, \dots, v_p)$ in $U_a^-$ satisfying (10). By (11) the sequence $\mathbf{v} =: (v_1, \dots, v_n)$ is a basis of $U$.

To prove the converse, observe that by the previous part of the proof we find a nonsingular matrix $\mathbf{C}$ such that

$$\mathbf{v}^T = \mathbf{C} \cdot \mathbf{u}^T.$$

If we have a basis $\mathbf{v} = (v_1, \ldots, v_k)$ such that (10) is satisfied, then the sequence $\mathbf{u} = (u_1, \ldots, u_n)$ defined by

$$\mathbf{u}^T = \mathbf{C}^{-1} \cdot \mathbf{v}^T$$

is a basis of $U$ such that (9) holds. This finishes the proof of Lemma 4. $\square$

**Corollary 3.** *If $p$ is a prime number and $a \in \mathcal{A}_X^{(p),-}$ then $a(\mathcal{H}) = \mathcal{H}$ for some Hamel basis $\mathcal{H} \subset X$.*

*Proof.* Fix $a \in \mathcal{A}_X^{(p),-}$. We have $X = X_a^+ \oplus X_a^-$ by Lemma 1. Since $\dim_{\mathbb{Q}} X_a^- < \infty$, on account of Theorem 2 we obtain a decomposition

$$X = X_a^+ \oplus X_a^- = X_a^+ \oplus \bigoplus_{i=1}^{k} \mathbf{Z}_i(v_i, a),$$

where $\mathbf{Z}_i(v_i, a)$ are $(p-1)$-dimensional $a$-cyclic subspaces. Theorem 2 implies also that there exists a Hamel basis $\mathcal{B} \subset X$ and its partition $\mathcal{B} = \overline{\mathcal{B}} \cup \bigcup_{i=1}^{k} \mathcal{B}_i$ with $\mathcal{B}_i = (v_1^i, \ldots, v_{p-1}^i)$ such that $a = \overline{a} \oplus \bigoplus_{i=1}^{k} a_i$, where $\operatorname{span} \overline{\mathcal{B}} = X_a^+$ and $\overline{a} = a|_{X_a^+} = \mathrm{id}_{X_a^+}$. Moreover, for every $i \in \{1, \ldots, k\}$ the additive function $a_i$ is $(q_2, \mathcal{B}_i)$-induced, where $q_2(x) = \sum_{j=0}^{p-1} x^j$. This means that

$$\begin{aligned} a(v_j^i) &= v_{j+1}^i && \text{for } j \in \{1, \ldots, p-2\}, \\ a(v_{p-1}^i) &= -v_1^i - \cdots - v_{p-1}^i. \end{aligned}$$

Choose different elements $v_0^1, \ldots, v_0^k \in \overline{\mathcal{B}}$ and let $\overline{\mathcal{B}}' = \overline{\mathcal{B}} \setminus \{v_0^1, \ldots, v_0^k\}$. For every $i \in \{1, \ldots, k\}$ we put $U_i = \operatorname{span}(\{v_0^i\}) \oplus \operatorname{span} \mathcal{B}_i = \operatorname{span}\left(v_0^i, v_1^i, \ldots, v_{p-1}^i\right)$. Thus $U_i$ is a $p$-dimensional space over $\mathbb{Q}$ with the basis $(v_0^i, v_1^i, \ldots, v_{p-1}^i)$ such that

$$\begin{aligned} a(v_0^i) &= v_0^i, \\ a(v_j^i) &= v_{j+1}^i && \text{for } j \in \{1, \ldots, p-2\}, \\ a(v_{p-1}^i) &= -v_1^i - \cdots - v_{p-1}^i. \end{aligned}$$

On account of Lemma 4 we find an ordered basis $\mathcal{B}_i' = (u_1, \ldots, u_p)$ in the space $U_i$ such that (9) holds. Hence $a(\mathcal{B}_i') = \mathcal{B}_i'$. We also have

$$\begin{aligned} X = \operatorname{span} \mathcal{B} &= \operatorname{span} \overline{\mathcal{B}} \oplus \bigoplus_{i=1}^{k} \operatorname{span} \mathcal{B}_i \\ &= \operatorname{span} \overline{\mathcal{B}}' \oplus \operatorname{span}\left(\{v_0^1, \ldots, v_0^k\}\right) \oplus \bigoplus_{i=1}^{k} \operatorname{span} \mathcal{B}_i \\ &= \operatorname{span} \overline{\mathcal{B}}' \oplus \bigoplus_{i=1}^{k} \operatorname{span}\left(\{v_0^i\} \cup \mathcal{B}_i\right) = \operatorname{span} \overline{\mathcal{B}}' \oplus \bigoplus_{i=1}^{k} \operatorname{span} \mathcal{B}_i'. \end{aligned}$$

Thus $\mathcal{H} = \overline{\mathcal{B}}' \cup \bigcup_{i=1}^{k} \mathcal{B}_i'$ is a basis in $X$. Since $a|_{\overline{\mathcal{B}}} = \mathrm{id}_{\overline{\mathcal{B}}}$, $a(\overline{\mathcal{B}}') = \overline{\mathcal{B}}'$ and $a(\mathcal{H}) = \mathcal{H}$. $\square$

## 6. Density

Let $X$ be a non-trivial real topological vector space and consider the topological space $X^X$ of all functions from $X$ into $X$, equipped with the Tychonoff topology. We will treat $\mathcal{A}_X$ as a topological space with the topology induced by $X^X$. In the proof of our main result we use, as it was in [1, Theorem 2] and [2, Theorem 4], the following lemma (see [1, Corollary 1]).

**Lemma 5.** *If for each $m \in \mathbb{N}$ and for every $h_1^1, \dots, h_1^m, h_2^1, \dots, h_2^m \in X$ being linearly independent over $\mathbb{Q}$, there exists $a \in \mathcal{D} \subset \mathcal{A}_X$ such that*

$$a(h_1^k) = h_2^k \quad \textit{for every} \quad k \in \{1, \dots, m\},$$

*then $\mathcal{D}$ is dense in $\mathcal{A}_X$.*

We prove

**Theorem 3.** *Let*

$$\begin{aligned} \mathcal{A}_1 = \{a \in \mathcal{A}_X^{(n)} : & \ a \textit{ is discontinuous and } a(\mathcal{H}) \setminus \mathcal{H} \neq \emptyset \textit{ for every infinite} \\ & \textit{set } \mathcal{H} \subset X \textit{ of vectors linearly independent over } \mathbb{Q}\} \\ \supset \mathcal{A}_X^{(p),+} & \quad \textit{for every prime number } p|n, \\ \mathcal{A}_2 = \{a \in \mathcal{A}_X^{(n)} : & \ a \textit{ is discontinuous and } a(\mathcal{H}) = \mathcal{H} \textit{ for some Hamel} \\ & \textit{basis } \mathcal{H} \subset X\} \\ \supset \mathcal{A}_X^{(p),-} & \quad \textit{for every prime number } p|n. \end{aligned}$$

*The sets $\mathcal{A}_X^{(p),+}$ and $\mathcal{A}_X^{(p),-}$, and consequently $\mathcal{A}_1$ and $\mathcal{A}_2$, are dense in the space $\mathcal{A}_X$.*

*Proof.* Fix a prime number $p|n$. Clearly $\mathcal{A}_X^{(p)} \subset \mathcal{A}_X^{(n)}$. On account of Theorem 1 and Corollaries 2 and 3 we have $\mathcal{A}_X^{(p),+} \subset \mathcal{A}_1$ and $\mathcal{A}_X^{(p),-} \subset \mathcal{A}_2$. To finish the proof we need to show the density of classes $\mathcal{A}_X^{(p),+}$ and $\mathcal{A}_X^{(p),-}$ in the space $\mathcal{A}_X$.

Fix $p \in \mathcal{P}$, $m \in \mathbb{N}$ and let $\mathbf{v} = \{h_1^1, \dots, h_1^m, h_2^1, \dots, h_2^m\} \subset X$ be linearly independent over $\mathbb{Q}$. Let $\mathcal{B} \subset X$ be an extension of $\mathbf{v}$ to a Hamel basis of $X$. Further, let $\mathcal{B}_i \subset \mathcal{B}$ for $i \in \{1, \dots, m\}$ be chosen in such a way that $\mathcal{B}_i = (h_1^i, h_2^i, \dots, h_p^i)$ (only the first two elements of every ordered basis $\mathcal{B}_i$ belong to each $\mathbf{v}$). We will proceed in each case separately.

*Case* $\mathcal{A}_X^{(p),-}$. Let $\overline{\mathcal{B}} := \mathcal{B} \setminus \bigcup_{i=1}^m \mathcal{B}_i$. Define $a \in \mathcal{A}_X$ as follows. Let $\overline{a} = \mathrm{id}_{\operatorname{span} \overline{\mathcal{B}}}$ and let $a_i$ be $(q, \mathcal{B}_i)$-induced on $\operatorname{span} \mathcal{B}_i$ for every $i \in \{1, \dots, m\}$, where $q(x) = x^p - 1$. Put $a = \overline{a} \oplus \bigoplus_{i=1}^m a_i$. This means that

$$\begin{aligned} a(h_j^i) &= h_{j+1}^i \quad \text{for } j \in \{1, \dots, p-1\}, \\ a(h_p^i) &= h_1^i. \end{aligned}$$

Note that in particular we have

$$a(h_1^i) = h_2^i \qquad \text{for } i \in \{1, \dots, m\}. \tag{12}$$

Fix $i \in \{1, \ldots, m\}$. On account of Lemma 4, for every $p$-dimensional space $\operatorname{span} \mathcal{B}_i$ we find a basis $\overline{\mathcal{B}}_i' = (v_0^i, v_1^i, \ldots, v_{p-1}^i)$ such that

$$\begin{aligned} a(v_0^i) &= v_0^i, \\ a(v_j^i) &= v_{j+1}^i \qquad\qquad \text{for } j \in \{1, \ldots, p-1\}, \\ a(v_p^i) &= -v_1^i - \cdots - v_p^i. \end{aligned}$$

Let $\mathcal{B}_i' = (v_1^i, \ldots, v_{p-1}^i)$ and $\overline{\mathcal{B}}' = \overline{\mathcal{B}} \cup \{v_0^1, \ldots, v_0^m\}$. Then $a|_{\operatorname{span} \mathcal{B}_i'}$ is $(q_2, \mathcal{B}_i')$-induced with $q_2(x) = \sum_{j=0}^{p-1} x^j$ and $a|_{\operatorname{span} \overline{\mathcal{B}}'} = \mathrm{id}_{\operatorname{span} \overline{\mathcal{B}}'}$. Since

$$\begin{aligned} X &= \operatorname{span} \overline{\mathcal{B}} \oplus \bigoplus_{i=1}^m \operatorname{span} \mathcal{B}_i = \operatorname{span} \overline{\mathcal{B}} \oplus \bigoplus_{i=1}^m \left( \operatorname{span} \{v_0^i\} \oplus \operatorname{span} \overline{\mathcal{B}}_i' \right) \\ &= \left( \operatorname{span} \overline{\mathcal{B}} \oplus \operatorname{span} \left( \{v_0^1, \ldots, v_0^m\} \right) \right) \bigoplus_{i=1}^m \operatorname{span} \overline{\mathcal{B}}_i' = \operatorname{span} \overline{\mathcal{B}}' \oplus \bigoplus_{i=1}^m \operatorname{span} \mathcal{B}_i', \end{aligned}$$

we see that $\mathcal{B}' = \overline{\mathcal{B}}' \cup \bigcup_{i=1}^m \mathcal{B}_i'$ is a Hamel basis of $X$. Moreover, $X_a^+ = \operatorname{span} \overline{\mathcal{B}}'$ and $X_a^- = \operatorname{span} \left( \bigcup_{i=1}^m \mathcal{B}_i' \right)$. Hence $\dim_{\mathbb{Q}} X_a^- < \infty$ and we get $a \in \mathcal{A}_X^{(p),-}$. Since $m$ was chosen arbitrarily, (12) and Lemma 5 imply that $\mathcal{A}_X^{(p),-}$ is dense in $\mathcal{A}_X$.

*Case* $\mathcal{A}_X^{(p),+}$. Let $(\mathcal{B}_k)_{k \in K}$ with $\mathcal{B}_k = (h_1^k, \ldots, h_{p-1}^k)$ be a $(p-1)$-partition of $\mathcal{B} \setminus \bigcup_{i=1}^m \mathcal{B}_i$. Define $a \in \mathcal{A}_X$ as follows. Let $a_i$ be $(q, \mathcal{B}_i)$-induced on $\operatorname{span} \mathcal{B}_i$ for $i \in \{1, \ldots, m\}$, where $q(x) = x^p - 1$ and let $a_j$ be $(q_2, \mathcal{B}_k)$-induced on $\operatorname{span} \mathcal{B}_j$ for $k \in K$, where $q_2(x) = \sum_{j=0}^{p-1} x^j$. Define $a = \bigoplus_{i=1}^m a_i \oplus \bigoplus_{k \in K} a_k$. This means that

$$\begin{cases} a(h_j^i) = h_{j+1}^i \quad \text{for } j \in \{1, \ldots, p-1\}, \\ a(h_p^i) = h_1^i, \end{cases} \qquad \text{if } i \in \{1, \ldots, m\},$$

$$\begin{cases} a(h_j^k) = h_{j+1}^k \qquad \text{for } j \in \{1, \ldots, p-2\}, \\ a(h_{p-1}^k) = -h_1^k - \cdots - h_{p-1}^k, \end{cases} \qquad \text{if } k \in K.$$

In particular we have

$$a(h_1^i) = h_2^i \quad \text{for } i \in \{1, \ldots, m\}. \tag{13}$$

Fix $i \in \{1, \ldots, m\}$. By Lemma 4 we find for every $p$-dimensional space $\operatorname{span} \mathcal{B}_i$ a basis $\overline{\mathcal{B}}_i' = (v_0^i, v_1^i, \ldots, v_{p-1}^i)$ such that

$$\begin{aligned} a(v_0^i) &= v_0^i, \\ a(v_j^i) &= v_{j+1}^i \qquad\qquad \text{for } j \in \{1, \ldots, p-1\}, \\ a(v_p^i) &= -v_1^i - \cdots - v_p^i. \end{aligned}$$

Let $\overline{\mathcal{B}} = (v_0^1, \ldots, v_0^m)$, $\mathcal{B}_i' = (v_1^i, \ldots, v_{p-1}^i)$ and $\overline{\mathcal{B}}' = \bigcup_{k \in K} \mathcal{B}_k \cup \bigcup_{i=1}^m \mathcal{B}_i'$. Then $a|_{\operatorname{span} \overline{\mathcal{B}}} = \mathrm{id}_{\operatorname{span} \overline{\mathcal{B}}}$, $a|_{\operatorname{span} \mathcal{B}_i'}$ is $(q_2, \mathcal{B}_i')$-induced for $i \in \{1, \ldots, m\}$ and $a|_{\operatorname{span} \mathcal{B}_k}$ is $(q_2, \mathcal{B}_k)$-induced for every $k \in K$. Since

$$
\begin{aligned}
X &= \bigoplus_{i=1}^{m} \operatorname{span} \mathcal{B}_i \oplus \bigoplus_{k\in K} \operatorname{span} \mathcal{B}_k = \bigoplus_{i=1}^{m} \operatorname{span} \left(\{v_0^i\} \cup \mathcal{B}_i'\right) \oplus \bigoplus_{k\in K} \operatorname{span} \mathcal{B}_k \\
&= \operatorname{span} \overline{\mathcal{B}} \oplus \bigoplus_{i=1}^{m} \operatorname{span} \mathcal{B}_i' \oplus \bigoplus_{k\in K} \operatorname{span} \mathcal{B}_k
\end{aligned}
$$

we see that $\mathcal{B}' = \overline{\mathcal{B}} \cup \bigcup_{i=1}^{m} \mathcal{B}_i' \cup \bigcup_{k\in K} \mathcal{B}_k$ is a Hamel basis of $X$. Finally, $X_a^+ = \operatorname{span} \overline{\mathcal{B}}$ and $X_a^- = \operatorname{span} \left(\bigcup_{i=1}^{m} \mathcal{B}_i' \cup \bigcup_{k\in K} \mathcal{B}_k\right)$, hence $\dim_{\mathbb{Q}} X_a^+ < \infty$ and we get $a \in \mathcal{A}_X^{(p),+}$. Since $m$ was chosen arbitrarily, (13) and Lemma 5 imply that $\mathcal{A}_X^{(p),+}$ is dense in $\mathcal{A}_X$. □

## 7. Conclusions

We have proved in Theorem 3 that the families $\mathcal{A}_1$ and $\mathcal{A}_2$ are dense in $\mathcal{A}_X$. The following question arises how big are they in the topological sense. We thus ask whether $\mathcal{A}_1$ and $\mathcal{A}_2$ are of first category in $\mathcal{A}_X$ or not. We may also consider a more general problem for algebraic operators, i.e. for additive functions satisfying

$$\alpha_0 \cdot \mathrm{id} + \sum_{j=1}^{k-1} \alpha_j \cdot a^j + a^k = 0.$$

Is it possible to prove similar results as Theorem 3 in this case?



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Received: March 30, 2015
Revised: August 23, 2015